

Reversible part of a quantum dynamical system

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Abstract

In this work a quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ is constituted by a von Neumann algebra \mathfrak{M} , by a unital Schwartz map $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$ and by a Φ -invariant normal faithful state φ on \mathfrak{M} . We prove that the ergodic properties of a quantum dynamical system, are determined by its reversible part $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$. It is constituted by a von Neumann sub-algebra \mathfrak{D}_∞ of \mathfrak{M} by an automorphism Φ_∞ and a normal state φ_∞ , the restrictions of Φ and φ on \mathfrak{D}_∞ respectively. Moreover, if \mathfrak{D}_∞ is a trivial algebra then the quantum dynamical system is ergodic. Furthermore we will give some properties of the reversible part of quantum dynamical system, in particular, we will study its relations with the canonical decomposition of Nagy-Fojas of linear contraction related to the quantum dynamical system.

1 Preliminaries and notations

We consider a pair (\mathfrak{M}, Φ) constituted by a von Neumann algebra \mathfrak{M} and a unital Schwartz map $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$ *i.e.* a σ -continuous map with $\Phi(1) = 1$ which satisfies the inequality:

$$0 \leq \Phi(a^*)\Phi(a) \leq \Phi(a^*a) \quad a \in \mathfrak{M} \quad (1.1)$$

In this work the pair (\mathfrak{M}, Φ) will be called (discrete) *quantum process* and Φ the dynamics of the quantum process.

A normal state φ on \mathfrak{M} is a stationary state for the quantum process (\mathfrak{M}, Φ) if $\varphi(\Phi(a)) = \varphi(a)$ for all $a \in \mathfrak{M}$, while is of asymptotic equilibrium if $\Phi^n(a) \rightarrow \varphi(a)1$ as $n \rightarrow \infty$ in σ -topology *i.e.*

$$\lim_{n \rightarrow +\infty} \omega(\Phi^n(a)) = \omega(1)\varphi(a) \quad a \in \mathfrak{M} \quad \omega \in \mathfrak{M}_*$$

We denote with $\mathcal{B}(\mathcal{H})$ the C^* -algebra of bounded linear operator on Hilbert space \mathcal{H} and with s and σ respectively the ultrastrong operator topology and the ultraweakly operator topology on von Neumann algebra \mathfrak{M} while with \mathfrak{M}_* its predual. Furthermore, a normal map or a normal state are σ -continuous maps (see ref. [6]).

We define the *multiplicative domain* \mathfrak{D}_Φ of a Schwartz map (see definition 2.1.4 and proposition 2.1.6 of [24]) as follows:

$$\mathfrak{D}_\Phi = \{a \in \mathfrak{M} : \Phi(a^*a) = \Phi(a^*)\Phi(a) \text{ and } \Phi(aa^*) = \Phi(a)\Phi(a^*)\}$$

We recall that an element $a \in \mathfrak{D}_\Phi$ if and only if $\Phi(ax) = \Phi(a)\Phi(x)$ and $\Phi(xa) = \Phi(x)\Phi(a)$ for all $x \in \mathfrak{M}$. It follows that \mathfrak{D}_Φ is a von Neumann algebra, since it is a unital $*$ -algebra closed in the σ -topology.

A consequence of the Schwartz's inequality is the following remark:

Remark 1. If $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$ is a unital Schwartz map which admits an inverse $\Phi^{-1} : \mathfrak{M} \rightarrow \mathfrak{M}$ (i.e. a unital Schwartz map such that $\Phi(\Phi^{-1}(a)) = \Phi^{-1}(\Phi(a)) = a$ for all $a \in \mathfrak{M}$), then Φ is an automorphism.

If \mathfrak{D}_∞^+ is the following von Neumann algebras:

$$\mathfrak{D}_\infty^+ = \bigcap_{n \in \mathbb{N}} \mathfrak{D}_{\Phi^n} \quad (1.2)$$

then we have that $\Phi(\mathfrak{D}_\infty^+) \subset \mathfrak{D}_\infty^+$ and Φ restricted to \mathfrak{D}_∞^+ is a *-homomorphism, but it is not surjective map.

Moreover we have:

$$\mathfrak{D}_\infty^+ = \{a \in \mathfrak{D}_\Phi : \Phi^n(a) \in \mathfrak{D}_\Phi \text{ for all } n \in \mathbb{N}\}$$

We define the *multiplicative core* of Φ (see ref. [23]):

$$\mathcal{C}_\Phi = \bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+) \subset \mathfrak{D}_\infty^+$$

We have $\Phi(\mathcal{C}_\Phi) \subset \mathcal{C}_\Phi$.

Indeed $\Phi^{n+1}(\mathfrak{D}_\infty^+) \subset \Phi^n(\mathfrak{D}_\infty^+)$ for all $n \geq 0$ and

$$\Phi\left(\bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+)\right) \subset \bigcap_{n \in \mathbb{N}} \Phi(\Phi^n(\mathfrak{D}_\infty^+)) = \bigcap_{n \in \mathbb{N}} \Phi^{n+1}(\mathfrak{D}_\infty^+) = \bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+)$$

It is clear that the restriction of Φ to multiplicative core \mathcal{C}_Φ is a *-homomorphism and if Φ is an injective map on \mathfrak{D}_∞^+ , then we have $\Phi(\mathcal{C}_\Phi) = \mathcal{C}_\Phi$, so the restriction of Φ to the multiplicative core is *-automorphism.

Since Φ is a normal map and its restriction to \mathfrak{D}_∞^+ is a *-homomorphism, we have that the set $\Phi^n(\mathfrak{D}_\infty^+)$ is a von Neumann algebra (see e.g. [6]), therefore \mathcal{C}_Φ is a von Neumann algebra.

Let φ be a stationary state for the quantum processes (\mathfrak{M}, Φ) and $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ its GNS representation. It is well know (see e.g. [20]) that there is a unique linear contraction $U_{\Phi, \varphi}$ of $\mathfrak{B}(\mathcal{H}_\varphi)$ such that, for any $a \in \mathfrak{A}$, we have

$$U_{\Phi, \varphi} \pi_\varphi(a) \Omega_\varphi = \pi_\varphi(\Phi(a)) \Omega_\varphi \quad (1.3)$$

Furthermore if φ is a faithful state then there is a unital Schwartz map $\Phi_\bullet : \pi_\varphi(\mathfrak{M}) \rightarrow \pi_\varphi(\mathfrak{M})$ such that

$$\Phi_\bullet(A) \Omega_\varphi = U_{\Phi, \varphi} A \Omega_\varphi \quad A \in \pi_\varphi(\mathfrak{M}) \quad (1.4)$$

It is simple to prove the following statements on multiplicative domains of Schwartz maps:

Proposition 1. Let (\mathfrak{M}, Φ) be quantum processes and φ its faithful stationary state, we have:

- a] For each $d \in \mathfrak{D}_\Phi$ results $U_{\Phi, \varphi} \pi_\varphi(d) = \pi_\varphi(\Phi(d)) U_{\Phi, \varphi}$
- b] If $U_{\Phi, \varphi} \pi_\varphi(a) = \pi_\varphi(\Phi(a)) U_{\Phi, \varphi}$ then $\Phi(ax) = \Phi(a)\Phi(x)$ for all $x \in \mathfrak{M}$
- c] $U_{\Phi, \varphi}^* U_{\Phi, \varphi} \in \pi_\varphi(\mathfrak{D}_\Phi)'$ while $U_{\Phi, \varphi} U_{\Phi, \varphi}^* \in \pi_\varphi(\Phi(\mathfrak{D}_\Phi))'$

$d]$ $d \in \mathfrak{D}_\Phi$ if, and only if $\|U_{\Phi,\varphi}\pi_\varphi(d)\Omega_\varphi\| = \|\pi_\varphi(d)\Omega_\varphi\|$ and $\|U_{\Phi,\varphi}\pi_\varphi(d^*)\Omega_\varphi\| = \|\pi_\varphi(d^*)\Omega_\varphi\|$

Proof. It is straightforward \square

We observe that if Φ is a $*$ -homomorphism, then the contraction $U_{\Phi,\varphi}$ is an isometry on \mathcal{H}_φ .

Another trivial consequence of Schwartz's inequality and of the existence of a faithful stationary state for quantum process (\mathfrak{M}, Φ) , are the following relations:

$$\cdots \mathfrak{D}_{\Phi^n} \subset \mathfrak{D}_{\Phi^{n-1}} \subset \cdots \mathfrak{D}_{\Phi^2} \subset \mathfrak{D}_\Phi \subset \mathfrak{M} \quad (1.5)$$

for all natural numbers $n \in \mathbb{N}$.

Furthermore, since Φ is a injective map on \mathfrak{D}_∞^+ , its restricted to \mathcal{C}_Φ is a $*$ -automorphism. In fact, if $a \in \mathfrak{D}_\infty^+$ with $\Phi(a) = 0$, then we obtain

$$\varphi(\Phi(a^*)\Phi(a)) = \varphi(\Phi(a^*a)) = \varphi(a^*a) = 0$$

Let (\mathfrak{M}, Φ) be a quantum processes and φ its stationary state, we recall that the dynamics Φ admits a φ -adjoint if there is a normal unital Schwartz map $\Phi^\sharp : \mathfrak{M} \rightarrow \mathfrak{M}$ such that

$$\varphi(b\Phi(a)) = \varphi(\Phi^\sharp(b)a) \quad a, b \in \mathfrak{M}$$

We have the following conditions for the existence of a φ -adjointness of dynamics of quantum process (see proposition 3.3 in [20]):

Proposition 2. *Let (\mathfrak{M}, Φ) be a quantum process and φ its faithful stationary state. If $(\Delta_\varphi, J_\varphi)$ denote the modular operators associated with pair $(\pi_\varphi(\mathfrak{M}), \Omega_\varphi)$, then the following conditions are equivalent:*

1 - Φ commutes with the modular automorphism group $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ i.e.

$$\sigma_t^\varphi \circ \Phi_\bullet = \Phi_\bullet \circ \sigma_t^\varphi \quad t \in \mathbb{R}$$

2 - $U_{\Phi,\varphi}$ commutes with modular operators:

$$U_{\Phi,\varphi}\Delta_\varphi^{it} = \Delta_\varphi^{it}U_{\Phi,\varphi} \quad t \in \mathbb{R}$$

and

$$U_{\Phi,\varphi}J_\varphi = J_\varphi U_{\Phi,\varphi}$$

3 - Φ admits φ -adjoint Φ^\sharp .

A triple $(\mathfrak{M}, \Phi, \varphi)$ constituted by quantum processes (\mathfrak{M}, Φ) , by its normal faithful stationary state φ and with dynamics Φ which admits a φ -adjoint Φ^\sharp , will be called a quantum dynamical system.

2 Decomposition theorem

We consider a von Neumann algebra \mathfrak{M} and its faithful normal state φ and set with $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ the GNS representation of φ and with $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ its modular automorphism group.

Let \mathfrak{R} be a von Neumann subalgebra of \mathfrak{M} , we recall (see ref. [14]) that the φ -orthogonal of \mathfrak{R} is the set:

$$\mathfrak{R}^{\perp_\varphi} = \{a \in \mathfrak{M} : \varphi(a^*x) = 0 \text{ for all } x \in \mathfrak{R}\} \quad (2.1)$$

Furthermore, it is simple to prove that $\mathfrak{R}^{\perp_\varphi}$ is a closed linear space in the σ -topology with $\mathfrak{R}^{\perp_\varphi} \cap \mathfrak{R} = \{0\}$.

We observe that $\mathfrak{R}^{\perp_\varphi} \subset \ker \varphi$ and if $\mathfrak{R} = \mathbb{C}I$ then $\mathfrak{R}^{\perp_\varphi} = \ker \varphi$, where $\ker \varphi = \{a \in \mathfrak{M} : \varphi(a) = 0\}$.

Moreover if $y \in \mathfrak{R}$ and $d_\perp \in \mathfrak{R}^{\perp_\varphi}$ then $yd_\perp \in \mathfrak{R}^{\perp_\varphi}$ since

$$\varphi((yd_\perp)^*x) = \varphi(d_\perp^*y^*x) = 0 \quad x \in \mathfrak{R}$$

Theorem 1. *The von Neumann algebra \mathfrak{R} is invariant under modular automorphism group σ_t^φ if and only if both these conditions are fulfilled:*

- a - the set $\mathfrak{R}^{\perp_\varphi}$ is closed under the involution operation;*
- b - for any $a \in \mathfrak{M}$ there is a unique $a_\parallel \in \mathfrak{R}$ and $a_\perp \in \mathfrak{R}^{\perp_\varphi}$ such that $a = a_\parallel + a_\perp$.
In other words we have the following algebraic decomposition*

$$\mathfrak{M} = \mathfrak{R} \oplus \mathfrak{R}^{\perp_\varphi} \quad (2.2)$$

Proof. From Takesaki [25] we have $\sigma_t^\varphi(\pi_\varphi(\mathfrak{R})) \subset \pi_\varphi(\mathfrak{R})$ for all $t \in \mathbb{R}$ if, and only if there exist a normal conditional expectation $\mathcal{E} : \mathfrak{M} \rightarrow \mathfrak{R}$ such that $\varphi \circ \mathcal{E} = \varphi$.

Let \mathfrak{R} be invariant under modular automorphism group σ_t^φ , it is simple to prove that

$$\mathfrak{R}^{\perp_\varphi} = \{a \in \mathfrak{M} : \mathcal{E}(a) = 0\} \quad (2.3)$$

hence $\mathfrak{R}^{\perp_\varphi}$ is closed under the involution operation.

For any $a \in \mathfrak{M}$ we set $a_\perp = a - \mathcal{E}(a)$ and

$$\varphi(a_\perp^*x) = \varphi((a^* - \mathcal{E}(a^*))x) = \varphi(a^*x) - \varphi(\mathcal{E}(a^*)x) = \varphi(a^*x) - \varphi(\mathcal{E}(a^*x)) = 0$$

for all $x \in \mathfrak{R}$ hence $a_\perp \in \mathfrak{R}^{\perp_\varphi}$.

So for any $a \in \mathfrak{M}$ there exist a unique $a_\parallel \in \mathfrak{R}$ and $a_\perp \in \mathfrak{R}^{\perp_\varphi}$ such that $a = a_\parallel + a_\perp$ where we have set $a_\parallel = \mathcal{E}(a)$.

The uniqueness follows because if $a = 0$ then $a_\parallel = a_\perp = 0$.

Indeed we have

$$\varphi(a^*a) = \varphi(a_\parallel^*a_\parallel) + \varphi(a_\perp^*a_\perp) = 0$$

since $a_\parallel^*a_\perp$, and $a_\perp^*a_\parallel$ belong to $\mathfrak{R}^{\perp_\varphi}$ and φ is a faithful state.

For the vice-versa, if the set $\mathfrak{R}^{\perp_\varphi}$ is closed under the involution operation and $\mathfrak{M} = \mathfrak{R} \oplus \mathfrak{R}^{\perp_\varphi}$ then for any $a \in \mathfrak{M}$ there is a unique $a_\parallel \in \mathfrak{R}$ and $a_\perp \in \mathfrak{R}^{\perp_\varphi}$ such that $a = a_\parallel + a_\perp$.

The map $a \in \mathfrak{M} \rightarrow a_\parallel \in \mathfrak{R}$ is a projection of norm one (i.e. it satisfies $(1)_\parallel = 1$ and $((a)_\parallel)_\parallel = a_\parallel$ for all $a \in \mathfrak{M}$), for Tomiyama [26] it is a normal conditional expectation (see [15] for a modern review) and $\varphi(a) = \varphi(a_\parallel)$ for all $a \in \mathfrak{M}$. \square

We observe that if $\mathfrak{R}^{\perp\varphi}$ is a $*$ -algebra (without unit) then $\mathfrak{R}^{\perp\varphi} = \{0\}$ since $\varphi(a_{\perp}^* a_{\perp}) = 0$ for all $a_{\perp} \in \mathfrak{R}^{\perp\varphi}$ and φ is a faithful state.

Moreover, if p is a orthogonal projector of \mathfrak{M} then $p \notin \mathfrak{R}^{\perp\varphi}$.

We have the following remark:

If $a \in \mathfrak{M}$ with $a = a_{\parallel} + a_{\perp}$ where $a_{\parallel} \in \mathfrak{R}$ and $a_{\perp} \in \mathfrak{R}^{\perp\varphi}$, then

$$\|\pi_{\varphi}(a)\Omega_{\varphi}\|^2 = \|\pi_{\varphi}(a_{\parallel})\Omega_{\varphi}\|^2 + \|\pi_{\varphi}(a_{\perp})\Omega_{\varphi}\|^2 \quad (2.4)$$

Proposition 3. *Let \mathfrak{R} be a von Neumann algebra invariant under modular automorphism group σ_t^{φ} . If \mathcal{H}_o and \mathcal{K}_o are the closure of the linear space $\pi_{\varphi}(\mathfrak{R})\Omega_{\varphi}$ and of $\pi_{\varphi}(\mathfrak{R}^{\perp\varphi})\Omega_{\varphi}$ respectively, then*

$$\mathcal{H}_{\varphi} = \mathcal{H}_o \oplus \mathcal{K}_o$$

Moreover the orthogonal projection P_o on Hilbert space \mathcal{H}_o belongs to $\pi_{\varphi}(\mathfrak{R})'$.

Proof. We have that $\mathcal{K}_o \subset \mathcal{H}_o^{\perp}$ since for any $r_{\perp} \in \mathfrak{R}^{\perp\varphi}$ and $\psi_o \in \mathcal{H}_o$ we obtain:

$$\langle \pi_{\varphi}(r_{\perp})\Omega_{\varphi}, \psi_o \rangle = \lim_{\alpha \rightarrow \infty} \langle \pi_{\varphi}(r_{\perp})\Omega_{\varphi}, \pi_{\varphi}(r_{\alpha})\Omega_{\varphi} \rangle = \lim_{\alpha \rightarrow \infty} \varphi(r_{\perp}^* r_{\alpha}) = 0$$

where $\psi_o = \lim_{\alpha \rightarrow \infty} \pi_{\varphi}(r_{\alpha})\Omega_{\varphi}$ with $\{r_{\alpha}\}_{\alpha}$ net belongs to \mathfrak{R} .

Let $\psi \in \mathcal{H}_{\varphi}$ we can write

$$\psi = \lim_{\alpha \rightarrow \infty} \pi_{\varphi}(m_{\alpha})\Omega_{\varphi} = \lim_{\alpha \rightarrow \infty} (\pi_{\varphi}(r_{\alpha})\Omega_{\varphi} + \pi_{\varphi}((r_{\alpha\perp})\Omega_{\varphi}))$$

where $m_{\alpha} = r_{\alpha} + r_{\alpha\perp}$ for each α .

The net $\{\pi_{\varphi}(r_{\alpha})\Omega_{\varphi}\}$ has limit, since by the relation (2.4) for each $\epsilon \geq 0$ there is a index ν such that for $\alpha \geq \nu$ and $\beta \geq \nu$ we have the Cauchy relation:

$$\|\pi_{\varphi}(r_{\alpha})\Omega_{\varphi} - \pi_{\varphi}(r_{\beta})\Omega_{\varphi}\| \leq \|\pi_{\varphi}(m_{\alpha})\Omega_{\varphi} - \pi_{\varphi}(m_{\beta})\Omega_{\varphi}\| \leq \epsilon$$

It follows that there are $\psi_{\parallel} \in \mathcal{H}_o$ and $\psi_{\perp} \in \mathcal{K}_o$ such that

$$\psi = \lim_{\alpha \rightarrow \infty} \pi_{\varphi}(r_{\alpha})\Omega_{\varphi} + \lim_{\alpha \rightarrow \infty} \pi_{\varphi}((r_{\alpha\perp})\Omega_{\varphi}) = \psi_{\parallel} + \psi_{\perp} \in \mathcal{H}_o \oplus \mathcal{K}_o$$

It is simple to prove that $\pi_{\varphi}(\mathfrak{R})\mathcal{H}_o \subset \mathcal{H}_o$ therefore $P_o \in \pi_{\varphi}(\mathfrak{R})'$. □

We have the following proposition:

Proposition 4. *Let (\mathfrak{M}, Φ) be a quantum process and φ a normal faithful state on \mathfrak{M} . For any natural number $n \in \mathbb{N}$ we obtain:*

$$\mathfrak{M} = \mathfrak{D}_{\Phi^n} \oplus \mathfrak{D}_{\Phi^n}^{\perp\varphi} \quad (2.5)$$

and

$$\mathfrak{M} = \mathfrak{D}_{\infty}^+ \oplus \mathfrak{D}_{\infty}^{+\perp\varphi} \quad (2.6)$$

Furthermore, if φ is a stationary state for Φ , then

$$\mathfrak{M} = \mathcal{C}_{\Phi} \oplus \mathcal{C}_{\Phi}^{\perp\varphi} \quad (2.7)$$

and the restriction of Φ to \mathcal{C}_{Φ} is a $*$ -automorphism with $\Phi(\mathcal{C}_{\Phi}^{\perp\varphi}) \subset \mathcal{C}_{\Phi}^{\perp\varphi}$.

Proof. for any $d \in \mathfrak{D}_{\Phi^n}$ and natural number n we have:

$$\Phi_{\bullet}^n(\sigma_t^\varphi(\pi_\varphi(d)^*)\sigma_t^\varphi(\pi_\varphi(d))) = \Phi_{\bullet}^n(\sigma_t^\varphi(\pi_\varphi(d)^*))\Phi_{\bullet}^n(\sigma_t^\varphi(\pi_\varphi(d))).$$

since Φ commutes with our modular automorphism group σ_t^φ . It follows that $\sigma_t^\varphi(\pi_\varphi(\mathfrak{D}_{\Phi^n}))$ is included in $\pi_\varphi(\mathfrak{D}_{\Phi^n})$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

Let $b \in \mathcal{C}_\Phi$, we have that $\sigma_t^\varphi(\pi_\varphi(b)) \in \pi_\varphi(\mathcal{C}_\Phi)$ for all real number t .

In fact for each natural number n there exist a $x_n \in \mathfrak{D}_\infty^+$ such that $b = \Phi^n(x_n)$. We can write that

$$\sigma_t^\varphi(\pi_\varphi(b)) = \sigma_t^\varphi(\pi_\varphi(\Phi^n(x_n))) = \Phi_{\bullet}^n(\sigma_t^\varphi(\pi_\varphi(x_n)))$$

and by above relation $\sigma_t^\varphi(\pi_\varphi(x_n)) \in \pi_\varphi(\mathfrak{D}_\infty^+)$ for all natural number n . It follows that $\sigma_t^\varphi(\pi_\varphi(b)) \in \pi_\varphi(\Phi^n(\mathfrak{D}_\infty^+))$ for all natural number n .

Let $y \in C_\Phi^{\perp\varphi}$, since $\Phi(\mathcal{C}_\Phi) = \mathcal{C}_\Phi$ we have for any $c \in \mathcal{C}_\Phi$ that

$$\varphi(\Phi(y)c) = \varphi(\Phi(y)\Phi(c_o)) = \varphi(yc_o) = 0$$

where $c = \Phi(c_o)$ with $c_o \in \mathcal{C}_\Phi \subset \mathfrak{D}_\infty^+$. □

We consider a quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ with φ -adjoint Φ^\sharp . We set with \mathfrak{D}_∞ (or with $\mathfrak{D}_\infty(\Phi)$ when we have to highlight the map Φ), the following von Neumann Algebra

$$\mathfrak{D}_\infty = \bigcap_{k \in \mathbb{Z}} \mathfrak{D}_{\Phi_k} \tag{2.8}$$

where for each k integer we denote

$$\Phi_k = \begin{cases} \Phi^k & k \geq 0 \\ \Phi^{\sharp|k|} & k < 0 \end{cases}$$

while with \mathfrak{D}_{Φ_k} we have set the von Neumann algebra of the multiplicative domains of the dynamics Φ_k .

Following [22], for each $a, b \in \mathfrak{M}$ and integers k we define:

$$S_k(a, b) = \Phi_k(a^*b) - \Phi_k(a^*)\Phi_k(b) \in \mathfrak{M} \tag{2.9}$$

and we have these simple relations:

- a - $S_k(a, a) \geq 0$ for all $a \in \mathfrak{M}$ and integers k ;
- b - $S_k(a, b)^* = S_k(b, a)$ for all $a, b \in \mathfrak{M}$ and integers k ;
- c - $d \in \mathfrak{D}_\infty$ if, and only if $S_k(d, d) = S_k(d^*, d^*) = 0$ for all integers k ;
- d - $d \in \mathfrak{D}_\infty$ if, and only if $\varphi(S_k(d, d)) = \varphi(S_k(d^*, d^*)) = 0$ for all integers k ;
- e - The map $a, b \in \mathfrak{M} \rightarrow \varphi(S_k(a, b))$ for all integers k , is a sesquilinear form, hence

$$|\varphi(S_k(a, b))|^2 \leq \varphi(S_k(a, a))\varphi(S_k(b, b)) \quad a, b \in \mathfrak{M}$$

We observe that $\Phi(\mathfrak{D}_\infty) \subset \mathfrak{D}_\infty$ and $\Phi^\sharp(\mathfrak{D}_\infty) \subset \mathfrak{D}_\infty$. Indeed for each element $d \in \mathfrak{D}_\infty$ and integer k we have

$$\varphi(S_k(\Phi(d), \Phi(d))) = \varphi(S_{k+1}(d, d)) = 0$$

and

$$\varphi(S_k(\Phi^\sharp(d), \Phi^\sharp(d))) = \varphi(S_{k-1}(d, d)) = 0$$

Furthermore $d^* \in \mathfrak{D}_\infty$ thus we obtain also

$$\varphi(S_k(\Phi(d)^*, \Phi(d)^*)) = \varphi(S_k(\Phi^\sharp(d)^*, \Phi^\sharp(d)^*)) = 0$$

It follows that restriction of the map Φ at von Neumann algebra \mathfrak{D}_∞ it is a $*$ -automorphism where $\Phi(\Phi^\sharp(d)) = \Phi^\sharp(\Phi(d)) = d$ for all $d \in \mathfrak{D}_\infty$.

We summarize the results obtained in following statement:

Proposition 5. *Let $(\mathfrak{M}, \Phi, \varphi)$ be a quantum dynamical system. The map $\Phi_\infty : \mathfrak{D}_\infty \rightarrow \mathfrak{D}_\infty$ where $\Phi_\infty(d) = \Phi(d)$ for all $d \in \mathfrak{D}_\infty$, is a $*$ -automorphism of von Neumann algebra.*

Furthermore if there is a von Neumann subalgebra \mathfrak{B} of \mathfrak{M} such that the restriction of Φ to \mathfrak{B} is a $$ -automorphism, then we obtain $\mathfrak{B} \subset \mathfrak{D}_\infty$.*

We have a (maximal) reversible quantum dynamical systems $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$ where the normal state φ_∞ and the φ_∞ -adjoint Φ_∞^\sharp , are respectively the restriction of φ and Φ^\sharp to the von Neumann algebra \mathfrak{D}_∞ .

Proof. We prove that if the restriction of Φ to \mathfrak{B} is an automorphism, then $\mathfrak{B} \subset \mathfrak{D}_\infty$. In fact we have that $\mathfrak{B} \subset \mathfrak{D}_{\Phi^n}$ for all natural number n and if $\Psi : \mathfrak{B} \rightarrow \mathfrak{B}$ is the map such that $\Psi(\Phi(b)) = \Phi(\Psi(b)) = b$ for all $b \in \mathfrak{B}$, then $\Psi(b) = \Phi^\sharp(b)$, since

$$\varphi(a\Psi(b)) = \varphi(\Phi(a\Psi(b))) = \varphi(\Phi(a)\Phi(\Psi(b))) = \varphi(\Phi(a)b) = \varphi(a\Phi^\sharp(b))$$

for all $a \in \mathfrak{M}$. It follows that \mathfrak{B} is also Φ^\sharp -invariant, hence $\mathfrak{B} \subset \mathfrak{D}_{\Phi^n^\sharp}$ for all natural number n . \square

It is clear that \mathfrak{D}_∞ is Φ_k -invariant for all integers k and is invariant under automorphism group σ_t^φ and by previous decomposition theorem we can say that (see [4] theorem 6):

Proposition 6. *If $(\mathfrak{M}, \Phi, \varphi)$ is a quantum dynamical system, then there is a conditional expectation $\mathcal{E}_\infty : \mathfrak{M} \rightarrow \mathfrak{D}_\infty$ such that*

$$a - \varphi \circ \mathcal{E}_\infty = \varphi;$$

$$b - \mathfrak{D}_\infty^{\perp_\varphi} = \ker \mathcal{E}_\infty;$$

$$c - \mathfrak{M} = \mathfrak{D}_\infty \oplus \mathfrak{D}_\infty^{\perp_\varphi};$$

$$d - \Phi_k(\mathfrak{D}_\infty^{\perp_\varphi}) \subset \mathfrak{D}_\infty^{\perp_\varphi} \text{ for all integers } k;$$

$$e - \mathcal{E}_\infty(\Phi_k(a)) = \Phi_k(\mathcal{E}_\infty(a)) \text{ for all } a \in \mathfrak{M} \text{ and integer } k;$$

$f - \mathcal{H}_\varphi = \mathcal{H}_\infty \oplus \mathcal{K}_\infty$ where \mathcal{H}_∞ and \mathcal{K}_∞ denotes the linear closure of $\pi_\varphi(\mathfrak{D}_\infty)\Omega_\varphi$ and of $\pi_\varphi(\mathfrak{D}_\infty^\perp)\Omega_\varphi$ respectively.

Proof. The statements (a), (b) and (c) are simple consequence of theorem 1.

For the statement (d), if $d_\perp \in \mathfrak{D}_\infty^\perp$ then for any integer k and $x \in \mathfrak{D}_\infty$, we have:

$$\varphi(\Phi_k(d_\perp)^*x) = \varphi(d_\perp^* \Phi_{-k}(x)) = 0$$

since $\Phi_{-k}(x) \in \mathfrak{D}_\infty$.

For the statement (e), for any $a, b \in \mathfrak{M}$ we obtain

$$\begin{aligned} \varphi(b\mathcal{E}_\infty(\Phi_k(a))) &= \varphi((b_\parallel + b_\perp)\mathcal{E}_\infty(\Phi_k(a))) = \varphi(b_\parallel\mathcal{E}_\infty(\Phi_k(a))) = \varphi(\mathcal{E}_\infty(b_\parallel\Phi_k(a))) = \\ &= \varphi(b_\parallel\Phi_k(a)) = \varphi(\Phi_{-k}(b_\parallel)a) = \varphi(\mathcal{E}_\infty(\Phi_{-k}(b_\parallel)a)) = \\ &= \varphi(\mathcal{E}_\infty(\Phi_{-k}(b_\parallel)a)) = \varphi(\Phi_{-k}(b_\parallel)\mathcal{E}_\infty(a)) = \varphi(b_\parallel\Phi_k(\mathcal{E}_\infty(a))) = \\ &= \varphi((b_\parallel + b_\perp)\Phi_k(\mathcal{E}_\infty(a))) = \varphi(b\Phi_k(\mathcal{E}_\infty(a))) \end{aligned}$$

where we have write $b = b_\parallel + b_\perp$ with $b_\parallel = \mathcal{E}_\infty(b)$. □

The quantum dynamical system $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$ is called *the reversible part* of the quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$.

Furthermore a quantum dynamical system is called *completely irreversible* if $\mathfrak{D}_\infty = \mathbb{C}1$. In this case for all $a \in \mathfrak{M}$ we obtain $a = \varphi(a)1 + a_\perp$ and we can write

$$\mathfrak{M} = \mathbb{C}1 \oplus \ker \varphi$$

Let $(\mathfrak{M}, \Phi, \varphi)$ be a completely irreversible quantum dynamical system, if the von Neumann algebra \mathfrak{M} is not trivial then there is least a not trivial projector $P \in \mathfrak{M}$, such that

$$\varphi(P) - \varphi(P)^2 > 0 \tag{2.10}$$

In fact, we can write $P = \varphi(P)1 + P_\perp$ where $P_\perp \in \mathfrak{D}_\infty^\perp$ and $\varphi(P_\perp^2) = \varphi(P) - \varphi(P)^2$ because $P_\perp^2 + 2\varphi(P)P_\perp + \varphi(P)^2 1 = \varphi(P)1 + P_\perp$. Therefore if $\varphi(P) = \varphi(P)^2$, then $P_\perp = 0$.

In section 4 we will find the conditions when $\mathfrak{D}_\infty = \mathbb{C}1$ (see also [8] section 2 for the case $\mathfrak{D}_\infty^+ = \mathbb{C}1$).

We observe that if $\mathcal{A}(\mathcal{P})$ is the von Neumann algebra generated by the set of all orthogonal projections $p \in \mathfrak{M}$ such that $\Phi_k(p) = \Phi_k(p)^2$ for all integers k , then $\mathfrak{D}_\infty = \mathcal{A}(\mathcal{P})$ (see [7], corollary 2).

In the decoherence theory the set \mathfrak{D}_∞ is called algebra of effective observables of our quantum dynamical system (see e.g. [3]) and we underline that the previous theorem is a particular case of a more general theorem that is found in [16].

We observe that for all natural number n we obtain

$$\Phi^{\#n}(\Phi^n(d)) = d \quad d \in \mathfrak{D}_\infty^+$$

and

$$\Phi^n(\mathfrak{D}_\infty^+) \subset \mathfrak{D}_{\Phi^{\#n}}$$

We can say more:

Remark 2. *The algebra of effective observables is independent by the stationary state φ , since*

$$\mathfrak{D}_\infty = \mathcal{C}_\Phi$$

In fact we have that $\mathfrak{D}_\infty \subset \bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+)$ since $\mathfrak{D}_\infty \subset \mathfrak{D}_\infty^+$ and $\mathcal{C}_\Phi \subset \mathfrak{D}_\infty$ for theorem 5.

The next subsections are of the simple consequences of the previous propositions.

2.1 Ergodicity properties

In this subsection we prove that the ergodic properties of a quantum dynamical system depends on its reversible part, determined from the algebra the effective observables \mathfrak{D}_∞ .

We consider a quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ with φ -adjoint Φ^\sharp .

We recall that the quantum dynamical system is ergodic if per any $a, b \in \mathfrak{M}$ we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left[\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right] = 0$$

while it is weakly mixing if

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left| \varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right| = 0$$

We will use again the following notations $a_\parallel = \mathcal{E}_\infty(a)$ while $a_\perp = a - a_\parallel$ for all $a \in \mathfrak{M}$, where $\mathcal{E}_\infty : \mathfrak{M} \rightarrow \mathfrak{D}_\infty$ is the conditional expectation of decomposition theorem 6.

We have the following proposition:

Proposition 7. *The quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ is ergodic [weakly mixing] if, and only if the reversible quantum dynamical system $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$ is ergodic [weakly mixing].*

Proof. For any $a, b \in \mathfrak{M}$ we have

$$\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) = \varphi(a\Phi^k(b_\parallel)) + \varphi(a\Phi^k(b_\perp)) - \varphi(a_\parallel)\varphi(b_\parallel)$$

Moreover $\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varphi(a\Phi^k(b_\perp)) = 0$, because by relation (3.6) for every $a \in \mathfrak{M}$,

we have $\lim_{k \rightarrow \infty} \varphi(a\Phi^k(b_\perp)) = 0$, hence

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left[\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right] = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left[\varphi(a\Phi^k(b_\parallel)) - \varphi(a_\parallel)\varphi(b_\parallel) \right]$$

with $\varphi(a\Phi^k(b_\parallel)) = \varphi(a_\parallel\Phi^k(b_\parallel)) + \varphi(a_\perp\Phi^k(b_\parallel))$ and $\varphi(a_\perp\Phi^k(b_\parallel)) = 0$ since the element $a_\perp\Phi^k(b_\parallel) \in \mathfrak{D}_\infty^{\perp\varphi}$.

It follows that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left[\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right] = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left[\varphi_\infty(a_\parallel\Phi_\infty^k(b_\parallel)) - \varphi_\infty(a_\parallel)\varphi_\infty(b_\parallel) \right]$$

For the weakly mixing properties we have

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left| \varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right| = \\
& = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left| \varphi(a_{\parallel}\Phi^k(b_{\parallel})) + \varphi(a_{\perp}\Phi^k(b_{\parallel})) + \varphi(a\Phi^k(b_{\perp})) - \varphi(a)\varphi(b) \right| = \\
& = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left| \varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel}) + \varphi(a\Phi^k(b_{\perp})) \right|
\end{aligned}$$

Moreover

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi(a\Phi^k(b_{\perp}))| = 0, \quad a, b \in \mathfrak{M}$$

If our quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ is weakly ergodic then

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \left| \varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel}) \right| = 0$$

since

$$\left| |\varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel})| - |\varphi(a\Phi^k(b_{\perp}))| \right| \leq \left| \varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right|,$$

while if the reversible dynamical system $(\mathfrak{D}_{\infty}, \Phi_{\infty}, \varphi_{\infty})$ is weakly mixing, then our quantum dynamical system is weakly mixing since

$$\left| \varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) \right| \leq \left| \varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel}) \right| + \left| \varphi(a\Phi^k(b_{\perp})) \right|$$

□

2.2 Particular *-Banach algebra

Let $(\mathfrak{M}, \Phi, \varphi)$ be a quantum dynamical system and $\mathcal{E}_{\infty} : \mathfrak{M} \rightarrow \mathfrak{D}_{\infty}$ the map of proposition 6.

We can define in set \mathfrak{M} another frame of *-Banach algebra changing the product between elements of \mathfrak{M} . It is defined by

$$a \times b = a_{\parallel} b_{\parallel} + a_{\parallel} b_{\perp} + a_{\perp} b_{\parallel} \quad a, b \in \mathfrak{M} \quad (2.11)$$

where we have denoted with $a_{\parallel} = \mathcal{E}_{\infty}(a)$ and with $a_{\perp} = a - a_{\parallel}$ for all $a \in \mathfrak{M}$.

We observe again that $a_{\parallel} b_{\perp}, a_{\perp} b_{\parallel} \in \mathfrak{D}_{\infty}^{\perp \varphi}$ since $\mathcal{E}_{\infty}(a_{\parallel} b_{\perp}) = a_{\parallel} \mathcal{E}_{\infty}(b_{\perp}) = 0$ and $\mathcal{E}_{\infty}(a_{\perp} b_{\parallel}) = \mathcal{E}_{\infty}(a_{\perp}) b_{\parallel} = 0$. Moreover we have

$$a_{\perp} \times b_{\perp} = 0$$

The $(\mathfrak{M}, +, \times)$ is a Banach *-algebra with unit, since for any $a, b \in \mathfrak{M}$ we have:

$$\|a \times b\| \leq \|a\| \|b\|$$

We set with \mathfrak{M}^b this Banach *-algebra.

We note that \mathfrak{M}^b it is not a C*-algebra. In fact for any $d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$, $d_\perp \neq 0$ we have that its spectrum in \mathfrak{M}^b is $\sigma(d_\perp) \subset \{0\}$ while $\|d_\perp\| \neq 0$.

We observe that for any $a, b \in \mathfrak{M}$ we have:

$$\Phi(a \times b) = \Phi(a) \times \Phi(b)$$

It follows that $\Phi : \mathfrak{M}^b \rightarrow \mathfrak{M}^b$ is a *-homomorphism of Banach Algebra.

For φ -adjoint Φ^\sharp we have:

$$\varphi(a \times \Phi(b)) = \varphi(a_\parallel \Phi(b_\parallel)) = \varphi(\Phi^\sharp(a_\parallel) b_\parallel) = \varphi(\Phi^\sharp(a) \times b)$$

with $\Phi^\sharp : \mathfrak{M}^b \rightarrow \mathfrak{M}^b$ *-homomorphism of Banach Algebra.

Moreover $\varphi(a^* \times a) = \varphi(a_\parallel^* a_\parallel)$ hence if $\varphi(a^* \times a) = 0$ then $a_\parallel = 0$, so φ it is not a faithful state on \mathfrak{M}^b .

It is easily to prove that for any $a, b \in \mathfrak{M}^b$ we obtain $\varphi(a^* \times b^* \times b \times a) = \varphi(a_\parallel^* b_\parallel^* b_\parallel a_\parallel)$ it follows that

$$\varphi(a^* \times b^* \times b \times a) \leq \|b\| \varphi(a^* \times a)$$

and we can build the GNS representation $(\mathcal{H}_\varphi^b, \pi_\varphi^b, \Omega_\varphi^b)$ of the state φ on Banach *-algebra \mathfrak{M}^b that has the following properties [9]:

The representation $\pi_\varphi^b : \mathfrak{M}^b \rightarrow \mathfrak{B}(\mathcal{H}_\varphi^b)$ is a continuous map i.e. $\|\pi_\varphi^b(a)\| \leq \|a\|$ for all $a \in \mathfrak{M}^b$ while Ω_φ^b is a cyclic vector for *-algebra $\pi_\varphi^b(\mathfrak{M}^b)$ and

$$\varphi(a) = \langle \Omega_\varphi^b, \pi_\varphi^b(a) \Omega_\varphi^b \rangle_b \quad a \in \mathfrak{M}^b$$

Furthermore we have a unitary operator $U_\varphi^b : \mathcal{H}_\varphi^b \rightarrow \mathcal{H}_\varphi^b$ such that

$$\pi_\varphi^b(\Phi(a)) = U_\varphi^b \pi_\varphi^b(a) U_\varphi^{b*} \quad a \in \mathfrak{M}^b$$

since Φ and Φ^\sharp are *-homomorphism of Banach algebra and

$$U_\varphi^b \pi_\varphi^b(a) \pi_\varphi^b(b) \Omega_\varphi^b = \pi_\varphi^b(\Phi(a \times b)) \Omega_\varphi^b = \pi_\varphi^b(\Phi(a)) \pi_\varphi^b(\Phi(b)) \Omega_\varphi^b = \pi_\varphi^b(\Phi(a)) U_\varphi^b \pi_\varphi^b(b) \Omega_\varphi^b$$

The linear map $Z : \mathcal{H}_\varphi^b \rightarrow \mathcal{H}_\varphi$ as defined $Z \pi_\varphi^b(a) \Omega_\varphi^b = \pi_\varphi(\mathcal{E}_\infty(a)) \Omega_\varphi$ for all $a \in \mathfrak{M}$ it is an isometry with adjoint $Z^* \pi_\varphi(a) \Omega_\varphi = \pi_\varphi^b(\mathcal{E}_\infty(a)) \Omega_\varphi^b$ for all $a \in \mathfrak{M}^b$.

Furthermore we have $Z U_\varphi^{bn} = Z U_{\Phi, \varphi}^n$ for all natural number n .

2.3 Abelian algebra of effective observables

We will prove that for any quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ there is an abelian algebra $\mathcal{A} \subset \mathfrak{D}_\infty$ that contains the center $Z(\mathfrak{D}_\infty)$ of \mathfrak{D}_∞ and with $\Phi(\mathcal{A}) \subset \mathcal{A}$.

The question of the existence of an Abelian subalgebra which remains invariant under the action of a given quantum Markov semigroup are widely debated in [2] and [21].

We consider a discrete quantum process (\mathfrak{M}, Φ) with Φ a *-automorphism. We set with $\mathfrak{P}(\mathfrak{M})$ the pure states of \mathfrak{M} .

It is well know that if $\omega(a) = 0$ for all $\omega \in \mathfrak{P}(\mathfrak{M})$ then $a = 0$ (see e.g. [5]).

For any $\omega \in \mathfrak{P}(\mathfrak{M})$ with \mathfrak{D}_ω we set the multiplicative domain of the ucp-map $a \in \mathfrak{M} \rightarrow \omega(a)I \in \mathfrak{M}$, then

$$\mathfrak{D}_\omega = \{a \in \mathfrak{M} : \omega(a^*a) = \omega(a^*)\omega(a) \text{ and } \omega(aa^*) = \omega(a)\omega(a^*)\}$$

it is a von Neumann subalgebra of \mathfrak{M} .

Proposition 8. *The von Neumann algebra*

$$\mathcal{A} = \bigcap \{ \mathfrak{D}_\omega : \omega \in \mathfrak{P}(\mathfrak{M}) \}$$

is an abelian algebra with $\Phi(\mathcal{A}) \subset \mathcal{A}$. Furthermore for any stationary state φ of our quantum process (\mathfrak{M}, Φ) , there is a φ -invariant conditional expectation $\mathcal{E}_\varphi : \mathfrak{M} \rightarrow \mathcal{A}$ such that

$$\mathcal{E}_\varphi \circ \Phi = \Phi$$

Proof. If $a, b \in \mathcal{A}$, for any pure state ω of \mathfrak{M} we have $\omega(ab) = \omega(a)\omega(b) = \omega(ba)$, then $\omega(ab - ba) = 0$ and it follows that $ab - ba = 0$.

The von Neumann algebra \mathcal{A} is Φ -invariant $\Phi(\mathcal{A}) \subset \mathcal{A}$.

In fact $\omega \circ \Phi \in \mathfrak{P}(\mathfrak{M})$ for all $\omega \in \mathfrak{P}(\mathfrak{M})$ since Φ is a $*$ -automorphism. Then for any $a \in \mathcal{A}$ we have

$$\omega(\Phi(a^*)\Phi(a)) = \omega(\Phi(a^*a)) = \omega(\Phi(a^*))\omega(\Phi(a))$$

it follows that $\Phi(a) \in \mathcal{A}$.

Let $\{\sigma_\varphi^t\}_{t \in \mathbb{R}}$ be a modular group associate to GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ of φ . Since the state φ is normal and faithful we have $\pi_\varphi(\mathcal{A})'' = \pi_\varphi(\mathcal{A})$ and $\sigma_\varphi^t(\pi_\varphi(\mathcal{A})) \subset \pi_\varphi(\mathcal{A})$ for all $t \in \mathbb{R}$.

In fact for any $a \in \mathcal{A}$ we have

$$\omega(\sigma_\varphi^t(a^*)\sigma_\varphi^t(a)) = \omega(\sigma_\varphi^t(a^*a)) = \omega(\sigma_\varphi^t(a^*))\omega(\sigma_\varphi^t(a)) \quad \omega \in \mathfrak{P}(\mathfrak{M})$$

since σ_φ^t is a $*$ -automorphism so $\omega \circ \sigma_\varphi^t \in \mathfrak{P}(\mathfrak{M})$ for all real number t .

From Takesaki theorem [25] we have that there is a conditional expectation $\mathcal{E}_\varphi : \mathfrak{M} \rightarrow \mathcal{A}$ such that

$$\pi_\varphi(\mathcal{E}_\varphi(m)) = \nabla^* \pi_\varphi(m) \nabla \quad m \in \mathfrak{M}$$

where $\nabla : \overline{\pi_\varphi(\mathcal{A})\Omega_\varphi} \rightarrow \mathcal{H}_\varphi$ is the embedding map (see also [1]). \square

We recall that any pure state is multiplicative on the center $Z(\mathfrak{M}) = \mathfrak{M} \cap \mathfrak{M}'$ of \mathfrak{M} (see [18]) so we have that $Z(\mathfrak{M}) \subset \mathfrak{D}_\omega$ for all pure states ω and in abelian case $\mathcal{A} = \mathfrak{M}$.

Let $(\mathfrak{M}, \Phi, \varphi)$ be a quantum dynamical system, with dynamics Φ that admits φ -adjoint Φ^\sharp .

By the decomposition theorem we have a $*$ -automorphism $\Phi_\infty : \mathfrak{D}_\infty \rightarrow \mathfrak{D}_\infty$ with \mathfrak{D}_∞ von Neumann algebra, then by the previous proposition, we can say that there exist an abelian algebra $\mathcal{A} \subset \mathfrak{D}_\infty$ with $\Phi(\mathcal{A}) \subset \mathcal{A}$ getting the following commutative diagram

$$\begin{array}{ccccc} & \mathfrak{M} & \xrightarrow{\Phi} & \mathfrak{M} & \\ i_\infty & \uparrow & & \downarrow & \mathcal{E}_\infty \\ & \mathfrak{D}_\infty & \xrightarrow{\Phi_\infty} & \mathfrak{D}_\infty & \\ i_o & \uparrow & & \downarrow & \mathcal{E}_\varphi \\ & \mathcal{A} & \xrightarrow{\Phi_o} & \mathcal{A} & \end{array}$$

where i_∞ and i_o are the embeddig of \mathfrak{D}_∞ and \mathcal{A} respectively, while Φ_∞ and Φ_o are the restriction of Φ to \mathfrak{D}_∞ and \mathcal{A} respectively.

We observe that if the von Neumann algebra \mathfrak{M} is abelian then $\mathcal{A} = \mathfrak{D}_\infty$.

2.4 Dilation properties

We recall that a reversible quantum dynamical system $(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi})$, is said to be a dilation of the quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$, if it satisfies the following conditions:

There is $*$ -monomorphism $i : (\mathfrak{M}, \varphi) \rightarrow (\widehat{\mathfrak{M}}, \widehat{\varphi})$ and a completely positive map $\mathcal{E} : \widehat{\mathfrak{M}} \rightarrow \mathfrak{M}$ such that for each a belong to \mathfrak{M} and natural number n

$$\mathcal{E}(\widehat{\Phi}^n(i(a))) = \Phi^n((a))$$

We observe that for each a belong to \mathfrak{M} and X in $\widehat{\mathfrak{M}}$ we have:

$$\mathcal{E}(i(a)X) = a\mathcal{E}(X).$$

Indeed for each $b \in \mathfrak{M}$ we obtain:

$$\varphi(b\mathcal{E}(i(a)X)) = \varphi(i(b)i(a)X) = \varphi(i(ba)X) = \varphi(ba\mathcal{E}(X))$$

So, the ucp-map $\widehat{\mathcal{E}} = i \circ \mathcal{E}$ is a conditional expectation from $\widehat{\mathfrak{M}}$ onto $i(\mathfrak{M})$ which leave invariant a faithful normal state. The existence of such map which characterize the range of existence of a reversible dilation of a dynamical system, be derived from a theorem of Takesaki of [25].

We have a proposition that establish a link between the algebra of effective observable and reversible dilation.

Proposition 9. *If $(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi})$ is a dilation of quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ then*

$$\widehat{\Phi}(i(a)) = i(\Phi(a)) \quad \text{if, and only if} \quad a \in \mathfrak{D}_{\Phi}$$

Proof. We have $i(\Phi(a)^*) i(\Phi(a)) = \widehat{\Phi}(i(a)^*) \widehat{\Phi}(i(a))$ it follows that

$$\Phi(a^*) \Phi(a) = \mathcal{E}(i(\Phi(a)^* \Phi(a))) = \mathcal{E}(\widehat{\Phi}(i(a^* a))) = \Phi(a^* a).$$

For vice-versa, if $y = i(\Phi(a)) - \widehat{\Phi}(i(a))$ then we have

$$y^* y = i(\Phi(a^* a)) - \widehat{\Phi}(i(a^*) i(\Phi(a)) - i(\Phi(a^*)) \widehat{\Phi}(i(a)) + \widehat{\Phi}(i(a^* a)))$$

since $a \in \mathfrak{D}_{\Phi}$. It follows that $\mathcal{E}(y^* y) = 0$ with \mathcal{E} faithful map, then $y = 0$. \square

Let $\mathfrak{M} = \mathfrak{D}_{\infty} \oplus \mathfrak{D}_{\infty}^{\perp \varphi}$ be decomposition of theorem 1 of our quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ and $\mathcal{E}_{\infty} : \mathfrak{M} \rightarrow \mathfrak{D}_{\infty}$ the conditional expectation defined in proposition 6, we can say:

Remark 3. *For each $a \in \mathfrak{M}$ and integer k we have:*

$$\widehat{\Phi}^k(i(\mathcal{E}_{\infty}(a))) = i(\Phi_k(\mathcal{E}_{\infty}(a)))$$

We observe that

$$X \in i(\mathfrak{D}_{\infty})^{\perp \widehat{\varphi}} \quad \text{if, and only if} \quad \mathcal{E}(X) \in \mathfrak{D}_{\infty}^{\perp \varphi}$$

since $i(\mathfrak{D}_{\infty})^{\perp \widehat{\varphi}} = \{X \in \widehat{\mathfrak{M}} : \widehat{\varphi}(X^* i(d)) = 0 \quad \forall d \in \mathfrak{D}_{\infty}\}$ and $\widehat{\varphi}(X^* i(d)) = \varphi(\mathcal{E}(X^*) d)$ for all $d \in \mathfrak{D}_{\infty}$.

We can write the following algebraic decomposition of linear spaces:

$$\widehat{\mathfrak{M}} = i(\mathfrak{D}_{\infty}) \oplus i(\mathfrak{D}_{\infty})^{\perp \widehat{\varphi}}$$

and the ucp-map $\widehat{\mathcal{E}}_{\infty} = i \circ \mathcal{E}_{\infty} \circ \mathcal{E}$ is a conditional expectation from $\widehat{\mathfrak{M}}$ onto $i(\mathfrak{D}_{\infty})$.

3 Decomposition theorem and linear contractions

We would study the relations between canonical decomposition of Nagy-Fojas of linear contraction $U_{\Phi, \varphi}$ [19] and decomposition (c) of proposition 6 of dynamical system $(\mathfrak{M}, \Phi, \varphi)$.

We going to recall the main statements of these topics.

A contraction T on the Hilbert space \mathcal{H} is called *completely non-unitary* if for no non zero reducing subspace \mathcal{K} for T is $T|_{\mathcal{K}}$ a unitary operator, where $T|_{\mathcal{K}}$ is the restriction of contraction T on the Hilbert space \mathcal{K} .

We set with $D_T = \sqrt{I - T^*T}$ the defect operator of the contraction T and it is well know that

$$TD_T = D_{T^*}T$$

Moreover $\|T\psi\| = \|\psi\|$ if, and only if $D_T\psi = 0$.

We consider the following Hilbert subspace of \mathcal{H} :

$$\mathcal{H}_0 = \{\psi \in \mathcal{H} : \|T^n\psi\| = \|\psi\| = \|T^{*n}\psi\| \text{ for all } n \in \mathbb{N}\} \quad (3.1)$$

It is trivial show that $T^n\mathcal{H}_0 = \mathcal{H}_0$ and $T^{*n}\mathcal{H}_0 = \mathcal{H}_0$ for all natural number n .

We have the following canonical decomposition (see [19]):

Theorem 2 (Sz-Nagy and Fojas). *To every contraction T on \mathcal{H} there corresponds a uniquely determined decomposition of \mathcal{H} into a orthogonal sum of two subspace reducing T we say $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that $T_0 = T|_{\mathcal{H}_0}$ is unitary and $T_1 = T|_{\mathcal{H}_1}$ is c.n.u., where*

$$\mathcal{H}_0 = \bigcap_{k \in \mathbb{Z}} \ker(D_{T_k}) \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H}_0^\perp$$

with

$$T_k = \begin{cases} T^k & k \geq 0 \\ T^{*-k} & k < 0 \end{cases}$$

It is well know [19] that the linear operator $T_- = so - \lim_{n \rightarrow +\infty} T^{*n}T^n$ and $T_+ = so - \lim_{n \rightarrow +\infty} T^nT^{*n}$, there are in sense of strong operator (so) convergence.

After this brief detour on linear contractions we return to quantum dynamical systems $(\mathfrak{M}, \Phi, \varphi)$.

We set $V_- = so - \lim_{n \rightarrow +\infty} U_{\Phi, \varphi}^{*n} U_{\Phi, \varphi}^n$ and $V_+ = so - \lim_{n \rightarrow +\infty} U_{\Phi, \varphi}^n U_{\Phi, \varphi}^{*n}$, where $U_{\Phi, \varphi}$ is the contraction defined in (1.3).

It follows that for each $a, b \in \mathfrak{M}$ we obtain:

$$\lim_{n \rightarrow \pm\infty} \varphi(S_n(a, b)) = \langle \pi_\varphi(a) \Omega_\varphi, (I - V_\pm) \pi_\varphi(b) \Omega_\varphi \rangle \quad (3.2)$$

where $S_n(a, b)$ is given by (2.9).

We recall that by proposition 6 that for every integers k we obtain $\mathcal{H}_\varphi = \mathcal{H}_\infty \oplus \mathcal{K}_\infty$ with $U_k\mathcal{H}_\infty = \mathcal{H}_\infty$ and $U_k\mathcal{K}_\infty \subset \mathcal{K}_\infty$, where

$$U_k = \begin{cases} U_{\Phi, \varphi}^k & k \geq 0 \\ U_{\Phi, \varphi}^{*-k} & k < 0 \end{cases}$$

A simple consequences of proposition 1 is the following remark:
For any integers k we obtain

$$a \in \mathfrak{D}_{\Phi_k} \quad \text{if, and only if} \quad \pi_\varphi(a)\Omega_\varphi \in \ker(D_{U_k}) \text{ and } \pi_\varphi(a^*)\Omega_\varphi \in \ker(D_{U_k})$$

Therefore $\mathcal{H}_\infty \subset \mathcal{H}_0$ because

$$\pi_\varphi(\mathfrak{D}_\infty)\Omega_\varphi \subset \bigcap_{k \in \mathbb{Z}} \pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi \subset \bigcap_{k \in \mathbb{Z}} \ker(D_{U_k})$$

We observe that for each $a, b \in \mathfrak{M}$ and natural number k we have (see [11] theorem 3.1):

$$\lim_{n \rightarrow +\infty} \varphi(S_k(\Phi^n(a), b)) = 0 \quad (3.3)$$

Indeed for each natural numbers k and n , we obtain

$$\varphi(S_k(\Phi^n(a), \Phi^n(b))) = \varphi(S_{k+n}(a, b)) - \varphi(S_n(a, b))$$

and by the relation (3.2) result $\lim_{n \rightarrow +\infty} (\varphi(S_{k+n}(a, b)) - \varphi(S_n(a, b))) = 0$.

Furthermore, for each natural number k and $a, b \in \mathfrak{M}$ we have

$$|\varphi(S_k(\Phi^n(a), b))|^2 \leq \varphi(S_k(\Phi^n(a), \Phi^n(a)))\varphi(S_k(b, b))$$

it follows that $\lim_{n \rightarrow +\infty} \varphi(S_k(\Phi^n(a), b)) = 0$.

We have a well-known statement (see [12], [16] and [23]):

Proposition 10. *For all $a \in \mathfrak{M}$ any σ -limit point of the set $\{\Phi^k(a)\}_{k \in \mathbb{N}}$ belongs to the von Neumann algebra \mathfrak{D}_∞ . Moreover, for each d_\perp in $\mathfrak{D}_\infty^{\perp\varphi}$ we have:*

$$\lim_{k \rightarrow +\infty} \Phi^k(d_\perp) = 0 \quad \text{and} \quad \lim_{k \rightarrow +\infty} \Phi^{\sharp k}(d_\perp) = 0$$

where the limits are in σ -topology.

Proof. If y is a σ -limit point of $\{\Phi^n(a)\}_{n \in \mathbb{N}}$ then there exists a net $\{\Phi^{n_j}(a)\}_{j \in \mathbb{N}}$ such that $y = \lim_{j \rightarrow +\infty} \Phi^{n_j}(a)$ in σ -topology. Furthermore for each $b \in \mathfrak{M}$ we obtain

$$S_k(y, b) = \sigma - \lim_{j \rightarrow +\infty} [\Phi^k(\Phi^{n_j}(a)b) - \Phi^k(\Phi^{n_j}(a))\Phi^k(b)] = \sigma - \lim_{j \rightarrow +\infty} S_k(\Phi^{n_j}(a), b)$$

from (3.3) we obtain $\lim_{j \rightarrow +\infty} \varphi(S_k(\Phi^{n_j}(a), b)) = 0$ hence $\varphi(S_k(y, b)) = 0$.

It follows that $\varphi(S_k(y, y)) = 0$ and $S_k(y, y) = 0$.

We observe that the adjoint is σ -continuous, then we obtain $y^* = \lim_{j \rightarrow +\infty} \Phi^{n_j}(a^*)$, and repeating the previous steps we obtain $S_k(y^*, y^*) = 0$, hence $y \in \mathfrak{D}_\infty$.

For last statement we observe that for each natural number k result $\|\Phi^k(d_\perp)\| \leq \|d_\perp\|$ and since the unit ball of the von Neumann algebra \mathfrak{M} is σ -compact we have that there is a subnet such that $\Phi^{k_\alpha}(d_\perp) \rightarrow y \in \mathfrak{D}_\infty^{\perp\varphi}$ in σ -topology. From previous lemma we have that $y \in \mathfrak{D}_\infty \cap \mathfrak{D}_\infty^{\perp\varphi}$ it follows that $y = 0$. then it can only be $\lim_{k \rightarrow +\infty} \Phi^k(d_\perp) = 0$ in σ -topology. \square

We observed that the Hilbert space \mathcal{H}_∞ , the linear closure of $\pi_\varphi(\mathfrak{D}_\infty)\Omega_\varphi$ is contained in \mathcal{H}_0 . The next step is to understand when we have the equality of these two Hilbert spaces.

Let $(\mathfrak{M}, \Phi, \varphi)$ be the previous quantum dynamical system, we define, for each integer k , the unital Schwartz map $\tau_k : \mathfrak{M} \rightarrow \mathfrak{M}$ as

$$\tau_k = \Phi_{-k} \circ \Phi_k \quad k \in \mathbb{Z} \quad (3.4)$$

We have for any integer k that

- 1 - $\varphi \circ \tau_k = \varphi$
- 2 - $\tau_k = \tau_k^\sharp$, where τ_k^\sharp is the φ -adjoint of τ_k .

We obtain, for any integer k the dynamical system $\{\mathfrak{M}, \tau_k, \varphi\}$ with

$$\mathfrak{D}_\infty(\tau_k) = \bigcap_{j \geq 0} \mathfrak{D}(\tau_k^j)$$

where with $\mathfrak{D}(\tau_k^j)$ we have denote the multiplicative domains of map τ_k^j . From decomposition theorem 1, for any integer k we have:

$$\mathfrak{M} = \mathfrak{D}_\infty(\tau_k) \oplus \mathfrak{D}_\infty(\tau_k)^{\perp_\varphi}$$

and by the proposition 3

$$\mathcal{H}_\varphi = \mathcal{H}_{(k)} \oplus \mathcal{K}_{(k)}$$

where $\mathcal{H}_{(k)}$ and $\mathcal{K}_{(k)}$ are the closure of the linear space $\pi_\varphi(\mathfrak{D}_\infty(\tau_k))\Omega_\varphi$ and of $\pi_\varphi(\mathfrak{D}_\infty(\tau_k))^{\perp_\varphi}\Omega_\varphi$ respectively.

We have the following proposition:

Proposition 11. *If $\overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi}$ denotes the closure of linear space $\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi$ then we have*

$$\mathcal{H}_0 = \bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi}$$

where the \mathcal{H}_0 is the Hilbert space of Nagy decomposition of theorem 2.

Furthermore for any $a \in \mathfrak{M}$ and $\xi_0 \in \mathcal{H}_0$ and integr k we have

$$U_{\Phi, \varphi}^k \pi_\varphi(a) \xi_0 = \pi_\varphi(\Phi^k(a)) U_{\Phi, \varphi}^k \xi_0 \quad (3.5)$$

Proof. We have that $\mathfrak{D}(\tau_k) \subset \mathfrak{D}_{\Phi_k}$ for all integers k .

In fact if $a \in \mathfrak{D}(\tau_k)$ then

$$\begin{aligned} \varphi(\Phi_k(a^*a)) &= \varphi(a^*a) = \varphi(\tau_k(a^*a)) = \varphi(\tau_k(a^*)\tau_k(a)) = \varphi(\Phi_{-k}(\Phi_k(a)^*)\Phi_{-k}(\Phi_k(a))) \leq \\ &\leq \varphi(\Phi_{-k}(\Phi_k(a)^*\Phi_k(a))) = \varphi(\Phi_k(a^*)\Phi_k(a)) \leq \varphi(\Phi_k(a^*a)) \end{aligned}$$

It follows that $\varphi(S_k(a, a)) = 0$ for all integers k and in the same way proves that $\varphi(S_k(a^*, a^*)) = 0$ for all integers k .

We have proved that

$$\mathfrak{D}_\infty(\tau_k) = \bigcap_{j \in \mathbb{N}} \mathfrak{D}_{\tau_k^j} \subset \mathfrak{D}_{\tau_k} \subset \mathfrak{D}_{\Phi_k}$$

If $\xi_0 \in \mathcal{H}_0$ then for any k integer and natural number n we have $(U_{\Phi,\varphi}^{*k}, U_{\Phi,\varphi}^k)^n \xi_0 = \xi_0$ and for any $r_\perp \in \mathfrak{D}_\infty(\tau_k)^\perp$ we can write that

$$\langle \pi_\varphi(r_\perp) \Omega_\varphi, \xi_0 \rangle = \left\langle (U_{\Phi,\varphi}^{*k}, U_{\Phi,\varphi}^k)^n \pi_\varphi(r_\perp) \Omega_\varphi, \xi_0 \right\rangle = \langle \pi_\varphi(\tau_k^n(r_\perp)) \Omega_\varphi, \xi_0 \rangle$$

and

$$\lim_{n \rightarrow +\infty} \langle \pi_\varphi(\tau_k^n(r_\perp)) \Omega_\varphi, \xi_0 \rangle = 0 \quad k \in \mathbb{Z}$$

since $\tau_k^n(r_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in σ -topology.

It follows that $\mathcal{H}_0 \subset [\pi_\varphi(\mathfrak{D}_\infty(\tau_k)^\perp) \Omega_\varphi]^\perp = [\mathcal{K}_k]^\perp$.

Therefore for any integers k we obtain:

$$\mathcal{H}_0 \subset \mathcal{H}_{(k)} \subset \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k}) \Omega_\varphi} \implies \mathcal{H}_0 \subset \bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k}) \Omega_\varphi}$$

Let $\xi_0 \in \bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k}) \Omega_\varphi}$, for any integers k we have a net $d_{\alpha,k} \in \mathfrak{D}_{\Phi_k}$ such that $\pi_\varphi(d_{\alpha,k}) \Omega_\varphi \rightarrow \xi_0$ as $\alpha \rightarrow \infty$ and for $k \geq 0$ we obtain

$$U_{\Phi,\varphi}^{*k} U_{\Phi,\varphi}^k \xi_0 = U_{\Phi,\varphi}^{*k} U_{\Phi,\varphi}^k \lim_{\alpha} \pi_\varphi(d_{\alpha,k}) \Omega_\varphi = \lim_{\alpha} U_{\Phi,\varphi}^{*k} U_{\Phi,\varphi}^k \pi_\varphi(d_{\alpha,k}) \Omega_\varphi = \lim_{\alpha} \pi_\varphi(d_{\alpha,k}) \Omega_\varphi = \xi_0$$

in the same way for $k \geq 0$ we have $U_{\Phi,\varphi}^k U_{\Phi,\varphi}^{*k} \xi_0 = \xi_0$.

It follows that

$$\bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k}) \Omega_\varphi} \subset \mathcal{H}_0$$

The relation (3.5) is a straightforward. □

We observe that for any $a \in \mathfrak{M}$ and $d_\perp \in \mathfrak{D}_\infty^\perp$ we have

$$\lim_{n \rightarrow \infty} \varphi(a^* \Phi_n(d_\perp) a) = 0 \quad (3.6)$$

since for any $d_\perp \in \mathfrak{D}_\infty^\perp$ we obtain $\Phi_n(d_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in σ -topology.

From polarization identity we can say that

$$\lim_{n \rightarrow \infty} \varphi(a \Phi_n(d_\perp) b) = 0, \quad a, b \in \mathfrak{M}, \quad d_\perp \in \mathfrak{D}_\infty^\perp$$

and since U_φ is a contraction it follows that for any $\xi \in \mathcal{H}_\varphi$ and $\psi \in \mathcal{K}_\infty$ we have

$$\lim_{n \rightarrow \infty} \langle \xi, U_{\Phi,\varphi}^n \psi \rangle = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \langle \xi, U_{\Phi,\varphi}^{*n} \psi \rangle = 0 \quad (3.7)$$

We give a simple statement on the Hilbert spaces \mathcal{H}_∞ and \mathcal{H}_0 :

Proposition 12. *If $\Phi^n(d_\perp) \rightarrow 0$ [$\Phi^\sharp(d_\perp) \rightarrow 0$] as $n \rightarrow \infty$ in s -topology for all $d_\perp \in \mathfrak{D}_\infty^\perp$; then $\mathcal{H}_\infty = \mathcal{H}_0$ and $V_+ = P_\infty$ [$V_- = P_\infty$]*

Proof. We observe that for any $\psi \in \mathcal{K}_\infty$ result $\|U_{\Phi,\varphi}^n \psi\| \rightarrow 0$ as $n \rightarrow \infty$, because for any $k \in \mathbb{N}$ there is $d_k^\perp \in \mathfrak{D}_\infty^\perp$ such that $\|\psi - \pi_\varphi(d_k^\perp) \Omega_\varphi\| < 1/k$ and $U_{\Phi,\varphi}^n$ is a linear contraction so for all natural number n we obtain:

$$\|U_{\Phi,\varphi}^n \psi\| < \frac{1}{k} + \varphi(\Phi^n(d_k^\perp)^* \Phi^n(d_k^\perp))$$

If $\xi_0 \in \mathcal{H}_0$, we can write $\xi_0 = \xi_{\parallel} + \xi_{\perp}$ with $\xi_{\parallel} \in \mathcal{H}_{\infty}$ and $\xi_{\perp} \in \mathcal{K}_{\infty}$. Then $\xi_{\perp} = \xi_0 - \xi_{\parallel} \in \mathcal{H}_0$ therefore

$$\|\xi_{\parallel}\| + \|\xi_{\perp}\| = \|U_{\Phi,\varphi}^n \xi_0\| = \|U_{\Phi,\varphi}^n \xi_{\parallel} + U_{\Phi,\varphi}^n \xi_{\perp}\| = \|\xi_{\parallel}\| + \|U_{\Phi,\varphi}^n \xi_{\perp}\|$$

for all natural numbers n it follows that $\xi_{\perp} = 0$.

Moreover for any $\xi \in \mathcal{H}_{\varphi}$ we have $U_{\Phi,\varphi}^{n*} U_{\Phi,\varphi}^n \xi = \xi_0 + U_{\Phi,\varphi}^{n*} U_{\Phi,\varphi}^n \xi_1$ with $\xi_i \in \mathcal{H}_i$ for $i = 1, 2$ and $V_+ \xi = \xi_0$ since $\|U_{\Phi,\varphi}^n \xi_1\| \rightarrow 0$ as $n \rightarrow \infty$. \square

We conclude this section with a simple observation:

We recall that a dynamical system $\{\mathfrak{M}, \Phi, \omega\}$ is mixing if

$$\lim_{n \rightarrow \infty} \varphi(a \Phi^n(b)) = \varphi(a) \varphi(b), \quad a, b \in \mathfrak{M} \quad (3.8)$$

by the relation (3.6) we obtain that $\{\mathfrak{M}, \Phi, \omega\}$ is mixing if, and only if its reversible part $(\mathfrak{D}_{\infty}, \Phi_{\infty}, \varphi_{\infty})$ is mixing.

Furthermore, let $\{\mathfrak{M}, \Phi, \omega\}$ be a mixing Abelian dynamical system, then there is a measurable dynamics space (X, \mathcal{A}, μ, T) such that \mathfrak{D}_{∞} is isomorphic to the von Neumann algebra $L^{\infty}(X, \mathcal{A}, \mu)$ of the measurable bounded function on X . If the set X is a metric space and φ_{∞} is the unique stationary state of \mathfrak{D}_{∞} for the dynamics Φ_{∞} , then by the corollary 4.3 of [10] we have $\mathfrak{D}_{\infty} = \mathbb{C}1$.

4 Decomposition theorem and Cesaro mean

In this section we will study the link between the decomposition theorem 1 and some ergodic results which we recall briefly.

It is well known the following proposition (see e.g. [13] par. 9.1 and [17] proposition 2.3).

Proposition 13. *Let $\{\mathfrak{M}, \tau, \omega\}$ be a quantum dynamical system. We consider the Cesaro mean*

$$s_n = \frac{1}{n+1} \sum_{k=0}^n \tau^k,$$

Then, there is an ω -conditional expectation \mathcal{E} of \mathfrak{M} onto fixed point $\mathcal{F}(\tau) = \{a \in \mathfrak{M} : \tau(a) = a\}$ such that

$$\lim_{n \rightarrow \infty} \|\phi \circ s_n - \phi \circ \mathcal{E}\| = 0 \quad \phi \in \mathfrak{M}_*$$

A simple consequence of the previous proposition is the following remark:

Remark 4. *$\{\mathfrak{M}, \tau, \omega\}$ is ergodic if, and only if $\mathcal{F}(\tau) = \mathbb{C}1$*

Let $(\mathfrak{M}, \Phi, \varphi)$ be the previous quantum dynamical system, and $\tau_k : \mathfrak{M} \rightarrow \mathfrak{M}$ the Schwartz map defined in (3.4), we have a simple statement:

Proposition 14. *For each integer k we obtain:*

$$\mathcal{F}(\tau_k) = \mathfrak{D}_{\Phi_k}$$

Proof. Without loss of generality we assume $k = 1$ then $\tau_1 = \Phi^{\sharp} \circ \Phi$.

If $x \in \mathcal{F}(\tau_1)$ we can write $\varphi(\Phi(x^*)\Phi(x)) = \varphi(x^* \tau_1(x)) = \varphi(x^* x) = \varphi(\Phi(x^* x))$ then $x \in \mathfrak{D}_{\Phi}$. The converse is proved similarly. \square

Now let us ask when the algebra of effectives observables \mathfrak{D}_∞ is trivial (see also [8] proposition 15) .

Proposition 15. *If $\mathfrak{D}_\infty = \mathbb{C}1$ then the normal state φ is of asymptotic equilibrium and the quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ is ergodic.*

Proof. By decomposition theorem $\mathfrak{M} = \mathbb{C}1 \oplus \mathfrak{D}_\infty^{\perp\varphi}$ and for each $a \in \mathfrak{M}$ we have $a = \varphi(a)1 + a_\perp$ with $a_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$. It follows that

$$\Phi^n(a) = \varphi(a)1 + \Phi^n(a_\perp)$$

and $\Phi^n(a_\perp) \rightarrow 0$ in σ -top. □

We have a simple consequence of the previous propositions:

Corollary 1. *If the quantum dynamical system $\{\mathfrak{M}, \tau_k, \varphi\}$ is ergodic for some integer k , then $\mathfrak{D}_\infty = \mathbb{C}1$.*

Proof. If we have ergodicity then $\mathcal{F}(\tau_k) = \mathfrak{D}_{\Phi_k} = \mathbb{C}1$. □

Summarizing

$$\tau_1 \text{ ergodic} \implies \Phi \text{ completely irreversible} \implies \Phi \text{ ergodic}$$

We observe that if $(\mathfrak{M}, \Phi, \varphi)$ is a quantum dynamical system with Φ homomorphism, we have that $\tau_1 = \Phi^\# \circ \Phi = id$. Hence the dynamical system $\{\mathfrak{M}, \tau_1, \varphi\}$ is not ergodic (if φ is not multiplicative functional), while $(\mathfrak{M}, \Phi, \varphi)$ can be.

For each integer k we consider $S_{n,k} = \frac{1}{n+1} \sum_{j=0}^n \tau_k^j$.

By previous proposition 13 there is a positive map $\mathcal{E}_k : \mathfrak{M} \rightarrow \mathfrak{M}$ such that

$$\|\phi \circ S_{n,k} - \phi \circ \mathcal{E}_k\| \rightarrow 0 \quad \phi \in \mathfrak{M}_*$$

and \mathcal{E}_k is the conditional expectation related of von Neumann algebra \mathfrak{D}_{Φ_k} of theorem 1.

Therefore $\mathcal{E}_k : \mathfrak{M} \rightarrow \mathfrak{D}_{\Phi_k}$ and $\varphi \circ \mathcal{E}_k = \varphi$ for all integers number k .

Furthermore we have:

$$\mathcal{E}_h \circ \mathcal{E}_k = \mathcal{E}_k \quad k \geq h \geq 0$$

because by the relation 1.5 we have $\mathfrak{D}_{\Phi^k} \subset \mathfrak{D}_{\Phi^h}$ for all $k \geq h$.

For each $a \in \mathfrak{M}$ we have $\|\mathcal{E}_k(a)\| \leq \|a\|$ for all integers k and apply σ -compactness property for the bounded net $\{\mathcal{E}_k(a)\}_{k \in \mathbb{N}}$ of von Neumann algebra \mathfrak{M} , we obtain that there is at least one σ -limit point $\mathcal{E}_+(a)$, therefore there exist a net $\{\mathcal{E}_{n_\alpha}(a)\}_\alpha$ such that $\mathcal{E}_+(a) = \sigma - \lim_\alpha \mathcal{E}_{n_\alpha}(a)$.

We obtain that $\mathcal{E}_+(a) \in \mathfrak{D}_{\Phi^k}$ for all natural number k because for any $a \in \mathfrak{M}$ we have $\mathcal{E}_h(\mathcal{E}_{n_\alpha}(a)) = \mathcal{E}_{n_\alpha}(a)$ when $n_\alpha \geq h$ and since \mathcal{E}_h are normal maps follows that

$$\mathcal{E}_h(\mathcal{E}_+(a)) = \mathcal{E}_+(a)$$

for all natural number h .

Furthermore, for any $x \in \mathfrak{D}_\infty^+$ we have

$$\varphi(xa) = \lim_{\alpha \rightarrow \infty} \varphi(\mathcal{E}_{n_\alpha}(xa)) = \lim_{\alpha \rightarrow \infty} \varphi(x\mathcal{E}_{n_\alpha}(a)) = \varphi(x\mathcal{E}_+(a))$$

it follows that we have a unique σ -limit point $\mathcal{E}_+(a)$ for the net $\{\mathcal{E}_n(a)\}_{n \in \mathbb{N}}$.

Therefore we obtain a map $\mathcal{E}_+ : \mathfrak{M} \rightarrow \mathfrak{D}_\infty^+$.

Moreover $\mathcal{E}_{n_\alpha}(\mathcal{E}_+(a)) = \mathcal{E}_+(a)$ for all α , then $\mathcal{E}_+^2 = \mathcal{E}_+$ and for Tomiyama [26] the positive map \mathcal{E}_+ is a conditional expectation such that $\varphi \circ \mathcal{E}_+ = \varphi$, precisely it is the conditional expectation of relation 2.6.

We can say something more:

Proposition 16. *Let $\{\mathfrak{M}, \Lambda_k, \varphi\}_{k \in \mathbb{N}}$ be a family of quantum dynamical systems. We consider the contraction $V_k : \mathcal{H}_\varphi \rightarrow \mathcal{H}_\varphi$ defined in (1.3) related to Schwartz map Λ_k :*

$$V_k \pi_\varphi(a) \Omega_\varphi = \pi_\varphi(\Lambda_k(a)) \Omega_\varphi \quad a \in \mathfrak{M}$$

If $\|[V_k^ - V_h^*]\xi\| \rightarrow 0$ as $h, k \rightarrow \infty$ for all $\xi \in \mathcal{H}_\varphi$, then there is a unital positive map $\Lambda : \mathfrak{M} \rightarrow \mathfrak{M}$ such that*

$$\|\phi \circ \Lambda_k - \phi \circ \Lambda\| \rightarrow 0 \quad (4.1)$$

as $k \rightarrow \infty$ for any $\phi \in \mathfrak{M}_$ with*

$$\varphi(\Lambda(a^*)\Lambda(a)) \leq \varphi(a^*a) \quad a \in \mathfrak{M}$$

and $\varphi \circ \Lambda = \varphi$.

Proof. A simple consequence of proposition 1.1 of [17] □

For each natural number n , we consider the following Schwartz map:

$$Z_n = \frac{1}{2n+1} \sum_{k=-n}^n \tau_k$$

it is obvious that φ is a stationary state for Z_n with $\varphi(xZ_n(y)) = \varphi(Z_n(x)y)$ for all $x, y \in \mathfrak{M}$.

Moreover for each $a \in \mathfrak{M}$ we have:

$$\begin{aligned} \pi_\varphi(Z_n(a)) \Omega_\varphi &= \frac{1}{2n+1} \sum_{k=-n}^n \pi_\varphi(\tau_k(a)) \Omega_\varphi = \\ &= \frac{1}{2n+1} \sum_{k=0}^n U_{\Phi, \varphi}^{*k} U_{\Phi, \varphi}^n \pi_\varphi(a) \Omega_\varphi + \frac{1}{2n+1} \sum_{k=1}^n U_{\Phi, \varphi}^k U_{\Phi, \varphi}^{*k} \pi_\varphi(a) \Omega_\varphi \end{aligned}$$

and since $U_{\Phi, \varphi}^{*n} U_{\Phi, \varphi}^n \rightarrow V_+$ and $U_{\Phi, \varphi}^n U_{\Phi, \varphi}^{*n} \rightarrow V_-$ in strong operator topology, we obtain

$$\pi_\varphi(Z_n(a)) \Omega_\varphi \rightarrow \frac{1}{2}(V_+ + V_-) \pi_\varphi(a) \Omega_\varphi$$

It follows that from previous proposition that there is a φ invariant Schwartz map $Z : \mathfrak{M} \rightarrow \mathfrak{M}$ such that

$$\|\phi \circ Z_n - \phi \circ Z\| \rightarrow 0 \quad \phi \in \mathfrak{M}_*$$

and

$$\pi_\varphi(Z(a)) \Omega_\varphi = \frac{1}{2}(V_+ + V_-) \pi_\varphi(a) \Omega_\varphi$$

We consider the decomposition $\mathfrak{M} = \mathfrak{D}_\infty \oplus \mathfrak{D}_\infty^{\perp\varphi}$ for each $a = a_\parallel + a_\perp \in \mathfrak{M}$ result

$$Z(a_\parallel + a_\perp) = a_\parallel + Z(a_\perp)$$

with $Z(a_\perp) \in \mathfrak{D}_\infty^{\perp\varphi}$.

We observe that if $\Phi^n(d_\perp) \rightarrow 0$ and $\Phi^{\#n}(d_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in s -topology for all $d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$ (see proposition 12); then $Z(d_\perp) = 0$ for all $d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$ and we have a φ invariant Schwartz map $Z : \mathfrak{M} \rightarrow \mathfrak{D}_\infty$ such that

$$Z(xa) = xZ(a), \quad x \in \mathfrak{M}, a \in \mathfrak{D}_\infty$$

It follows that Z is the conditional expectation \mathcal{E}_∞ of proposition 6.

References

- [1] Accardi L. and Cecchini C.: *Conditional expectations on von Neumann algebras*, J. Funct. Anal. **45** 245-273 (1982).
- [2] Attal A. and Rebolledo R.: *Quantum stationary states and classical Markov semigroups*, Unpublished preprints.
- [3] Blanchard M. and Olkiewicz R.: *Decoherence induced transition from quantum to classical dynamics*, Phys. Rev. Lett. **90**, 010403 (2003).
- [4] Blanchard M. and Hellmich M.: *Decoherence in infinite quantum systems*, AIP Conf. Proc. 1469, 2 (2012).
- [5] Blackadar B.: *Operator algebra*, Springer (2006).
- [6] Bratteli O. and Robinson D.W.: *Operator algebras and quantum mechanics Vol.I*, Springer-Verlag (1979).
- [7] Carbone R., Sasso E. and Umanit  V.: *Decoherence for positive semigroups on $M_2(\mathbb{C})$* , J. Math. Phys. **52**, 032202 (2011).
- [8] Carbone R., Sasso E. and Umanit  V.: *Ergodic quantum Markov semigroups and decoherence*, J. Oper. Theory 72(2) 293–312 (2014).
- [9] Doran R.S. and Fell J.M.: *Representation of *-algebras , locally compact groups, and Banach *-algebras bundles, Vol. 1*, Academic press Inc. (1988).
- [10] Fidaleo F.: *On strong ergodic properties of quantum dynamical systems*, Infin. Dimens. Anal. Quantum Probab. Rel. Top. Vol. 12, No. 4 (2009).
- [11] Frigerio A.: *Stationary states of quantum dynamical semigroups*, Commun. Math. Phys. **63** 269-276 (1978).
- [12] Hellmich M.: *Decoherence in infinite quantum systems* , Phd thesis University of Bielefeld.
- [13] Krengel U.: *Ergodic theorems* , De Gruyter (1985).

- [14] Kummerer B.: *Markov dilation on W^* -algebras*, J. Funct. Anal. **63**, 139-177 (1985).
- [15] Li B.R.: *Introduction to operator algebras*, World Scientific (1992).
- [16] Lugiewicz P. and Olkiewicz R.: *Classical properties of Infinite quantum open systems*, Commun. Math. Phys. 239, 241259 (2003).
- [17] Mohari A.: *A mean ergodic theorem of an amenable group action*, Infin. Dimens. Anal. Quantum Probab. Rel. Top. Vol. 17, No. 1 (2014).
- [18] Moriyoshi H. and Natsume T.: *Operator algebras and geometry*, Amer. Math. Soc. Vol. 237 (2008),
- [19] Nagy B. Sz. - Foiaş C.: *Harmonic analysis of operators on Hilbert space*, Regional Conference Series in Mathematics, n.**19**, (1971).
- [20] Niculescu C., Ströh A. and Zsidó L.: *Non commutative extensions of classical and multiple recurrence theorems*, Operator Theory **50** 3-52 (2002).
- [21] Rebolledo R.: *Decoherence of quantum Markov semigroups*, Ann. I. H. Poincaré PR 41 349373 v(2005).
- [22] Robinson D.W.: *Strongly positive semigroups and faithful invariant states* - Commun. Math. Phys. 85, 129-142 (1982)
- [23] Stormer E.: *Multiplicative properties of positive maps*, Math. Scand. **100**, 184-192 (2007).
- [24] Stormer E.: *Positive linear maps of operator algebras*, Springer-Verlag (2013).
- [25] Takesaki M.: *Conditional expectations in von Neumann algebras*, J. Funct. Anal. **9** 306-321 (1972).
- [26] Tomiyama J.: *On the projection of norm one in W^* -algebras*, Proc. Japan Acad. **33** 608-612 (1957).